

VOLUME OF PARTIALLY HYPERBOLIC HORSESHOE-LIKE ATTRACTORS

ABBAS FAKHARI AND MOHAMMAD SOUFI

ABSTRACT. We prove that any partially hyperbolic horseshoe-like attractor of a C^1 -generic diffeomorphism has zero volume. By modification of Poincaré cross section of the geometric model of Lorenz attractor, we build a C^1 -diffeomorphism with a partially hyperbolic horseshoe-like attractor of positive volume.

1. INTRODUCTION

A hyperbolic set is a compact invariant set over which the tangent bundle splits into two invariant sub-bundles, one is contracting and the other one is expanding. The Lebesgue measure (volume) of hyperbolic sets is an interesting subject considered in many articles. The scenario was begun by the seminal works of Bowen in 70's. Bowen has proved in [5] that a hyperbolic attractor of positive volume of a C^2 -diffeomorphism does contains some stable and unstable manifolds. On the other hand, he showed in [4] the existence of a C^1 -diffeomorphism admitting a totally disconnected hyperbolic set of positive volume. The issue of the volume and the interior of a hyperbolic sets followed by many authors. For instance, it is shown in [2] that a transitive hyperbolic set which attracts a set with positive volume necessarily attracts a neighborhood of itself. It is also proven in [2] that there are no proper transitive hyperbolic sets with positive volume for diffeomorphisms whose differentiability is higher than one. In the context of volume preserving diffeomorphism, it is proved in [3], that a volume preserving diffeomorphism with a hyperbolic set of positive volume should be an Anosov diffeomorphism (see also [7]). The main point in this context is the saturation of an invariant set of positive volume. In light of the rich consequences on the volume of hyperbolic sets, a lot of ground work had been undertaken in a more general landscape which is partially hyperbolic. The tangent bundle over a partially hyperbolic set splits into bundles, one of them is hyperbolic and one is middle (nor contracting nor expanding). Alves and Pinheiro proved in [2] the nonexistence of horseshoe-like partially hyperbolic sets. Extending their result, Zhong has proved in [8] that the hyperbolic lamination of the limit set of a partially hyperbolic set Λ of a C^2 -diffeomorphism should be contained in Λ . The aim of this article is to study the volume of a partially hyperbolic sets from the C^1 -generic point of view. Using the fact of zero volume of a horseshoe-like partially hyperbolic set of a $C^{1+\alpha}$ diffeomorphism, we show the same fact for the horseshoe-like attractors of C^1 -generic partially hyperbolic attractors. Inspired by the example of Bowen, we build a C^1 flow whose

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time one map admits a horseshoe-like partially hyperbolic attractor of positive volume. Our construction is based on a modification of a classical Lorenz flow using the Bowen map.

We begin our context by recalling some essential notions. Let M be a Riemannian boundaryless manifold equipped with a smooth volume Leb and $f : M \rightarrow M$ be a diffeomorphism. If the derivative Df of f is Hölder continuous with exponent α then f is said to be a $C^{1+\alpha}$ -diffeomorphism. A closed invariant subset Λ is an *attractor* if there is an open set U containing Λ in such a way $f(\overline{U}) \subset U$ and $\bigcap_{n \in \mathbb{N}} f^n(U) = \Lambda$. An invariant set Λ admits a dominated splitting $E \oplus F$ over Λ if there is $0 < \lambda < 1$ such that

$$\|Df|_{E(x)}\| \cdot \|Df^{-1}|_{F(f(x))}\| \leq \lambda.$$

A dominated splitting $E^s \oplus F$ is partially hyperbolic if $\|Df|_{E^s}\| < \lambda$. It is known that if f has a partially hyperbolic set Λ then there is a neighborhood U of Λ and \mathcal{U} of f in C^r topology such that any $g \in \mathcal{U}$ admits a partially hyperbolic structure on the invariant set $\Lambda_g = \bigcap_{n \in \mathbb{Z}} g^n(U)$. We note that the set Λ_g is not necessarily closed to the initial set Λ in the Hausdorff metric.

For a partially hyperbolic invariant set Λ of f there is a local dynamical lamination W_δ^s integrating the direction E^s that is $T_x W_\delta^s(x) = E^s(x)$. Furthermore, for any sufficiently C^1 -close diffeomorphism g and any two points $x \in \Lambda$ and $y \in \Lambda_g$, the local stable manifold $W_\delta^s(x, g)$ are sufficiently closed to $W_\delta^s(x, f)$ in Hausdorff metric.

We say that a partially hyperbolic set Λ is *horseshoe-like* if it does not contain any local stable manifold of its points.

Theorem A. *There is a residual subset $\mathcal{R}_0 \subset \text{Diff}^1(M)$ such that for any $f \in \mathcal{R}_0$, if Λ is a partially hyperbolic horseshoe-like attractor of f then $\text{Leb}(\Lambda) = 0$.*

Our goal here is to complement our scenario by presenting a (non-generic) example of diffeomorphism in C^1 -topology exhibiting a horseshoe-like partially hyperbolic attractor of positive Lebesgue measure. Our approach is based on a modification of the classical geometric model for Lorenz attractor introduced firstly by [6]. Before, let us recall the notion of partial hyperbolicity for flows over a compact invariant set Λ of a vector field X , which is very similar to the previous one. A compact invariant set Λ of a vector field X is *partially hyperbolic* if there is an invariant splitting $E^s \oplus F$ such that for some fixed $0 < \lambda < 1$,

$$\|DX_t|_{E^s(x)}\| \cdot \|DX_{-t}|_{F(X_t(x))}\| < \lambda^t,$$

and it also holds $\|DX_t|_{E^s(x)}\| < \lambda^t$. We say that the sub-bundle F is volume expanding if $J_t|_F(x) \geq \exp^{\lambda t}$, where $J_t|_F(x)$ is the Jacobian of $X_t : F(x) \rightarrow F(X_t(x))$. In this setting, we say that Λ is partially hyperbolic with *volume expanding central direction*. The set Λ is a *singular-hyperbolic* set for X if all singularities of Λ are hyperbolic, and Λ is partially hyperbolic with volume expanding central direction.

Theorem B. *There is a C^1 -diffeomorphism f on a three dimensional manifold admitting a horseshoe-like partially hyperbolic attractor of positive volume. Mapping f is time one map of a flow with a singular hyperbolic attractor.*

2. C^1 -GENERIC ATTRACTORS

In [7], Xia introduced a simple dynamical density basis for general partially hyperbolic sets. Using the dynamical density points, he gave the following remarkable proposition leading to the saturation of invariant sets of positive Lebesgue measure in the context of conservativeness. The proposition holds for any $C^{1+\alpha}$ -diffeomorphism, whether it is conservative or not.

Proposition 1. *Let f be a $C^{1+\alpha}$ diffeomorphism and Λ be a partially hyperbolic subset of M of positive volume. For Lebesgue-a.e point $x \in \Lambda$, $\delta > 0$ and $\epsilon > 0$ there is k_0 depending only to x and ϵ and not to δ such that for any $k \geq k_0$,*

$$\mu^s(W_\delta^s(f^{-k}(x)) \cap \Lambda) \geq (1 - \epsilon)\mu^s(W_\delta^s(f^{-k}(x))),$$

where μ^s is the induced Lebesgue measure on the stable leaf.

Theorem 2. *Under the assumption of the proposition above, if y belongs to the α -limit set of a point $x \in \Lambda$, then*

$$W_\delta^s(y) \subset \Lambda$$

Proof. Let $f^{-n_k}(x) \rightarrow y$. Then by the continuity of the local stable manifolds we have $W_\delta^s(f^{-n_k}(x)) \rightarrow W_\delta^s(y)$. Now, by contradiction, suppose that $W_\delta^s(y) \not\subset \Lambda$. Hence, there is a point $z \in W_\delta^s(y) \setminus \Lambda$. Take $\delta_0 \leq \delta$ for which $B_{\delta_0}(z) \cap \Lambda = \emptyset$, where $B_{\delta_0}(z)$ is the δ_0 -neighborhood of z . Let $x_k \in W_\delta^s(f^{-n_k}(x))$ such that $x_k \rightarrow z$. For large k , $x_k \in B_{\delta_0/2}(z)$ and so,

$$B_{\delta_0/2}(x_k) \cap \Lambda = \emptyset.$$

In particular, $W_{\delta_0/2}^s(x_k) \cap \Lambda = \emptyset$, for large k . As $W_{\delta_0/2}^s(x_k) \subset W_\delta^s(f^{-n_k}(x))$, we have

$$\begin{aligned} (1 - \epsilon)\mu^s(W_\delta^s(f^{-n_k}(x))) &\leq \mu^s(W_\delta^s(f^{-n_k}(x)) \cap \Lambda) \leq \mu^s(W_\delta^s(f^{-n_k}(x)) \setminus W_{\delta_0/2}^s(x_k)) \\ &= \mu^s(W_\delta^s(f^{-n_k}(x))) - C\delta_0, \end{aligned}$$

which is a contradiction since k does not depend on δ . \square

As a direct consequence, we have an easy, yet essential, corollary.

Corollary 3. *Any horseshoe-like partially hyperbolic set of a $C^{1+\alpha}$ -diffeomorphism has zero Lebesgue measure.*

Proof of Theorem A. Let \mathfrak{B} be a countable open base for the topology of M and $\{V_n\}_{n \in \mathbb{N}}$ be the family of all finite union of the elements of \mathfrak{B} . Suppose that \mathcal{V}_n is the set of all $f \in \text{Diff}^1(M)$ such that $f(\overline{V_n}) \subset V_n$ and $\cap_{m \in \mathbb{N}} f^m(V_n)$ is partially hyperbolic. By the robustness of the partial hyperbolicity, the \mathcal{V}_n is an open set. Now, define $\mathcal{F}_n : \mathcal{V}_n \rightarrow 2^M$ by $g \mapsto A_n(g)$, where $A_n(g) = \cap_{m \in \mathbb{N}} g^m(V_n)$. It is not difficult to see that \mathcal{F}_n is an upper-semicontinuous. Hence, there is a residual subset \mathcal{R}_n of \mathcal{V}_n such that \mathcal{F}_n varies continuously on it. We note that by the regularity of Lebesgue measure, any $f \in \mathcal{R}_n$ is also a continuity point of the mapping $\mathcal{G}_n : \mathcal{V}_n \rightarrow \mathbb{R}$ given by $g \mapsto \text{Leb}(A_n(g))$. Now, for any n , put $\mathcal{H}_n = \text{Diff}^1(M) \setminus \overline{\mathcal{V}_n}$ and $\mathcal{R}'_n = \mathcal{H}_n \cup \mathcal{R}_n$. Then \mathcal{R}'_n is a residual subset of $\text{Diff}^1(M)$. Put $\mathcal{R} = \cap_{n \in \mathbb{N}} \mathcal{R}'_n$. Let $f \in \mathcal{R}$ has a

partially hyperbolic horseshoe-like attractor A and let $A = \bigcap_{n \in \mathbb{N}} f^n(U)$. Choose n_0 such that $A \subset V_{n_0} \subset U$. Then $A = A_{n_0}(f)$ and so $f \in \mathcal{V}_{n_0}$. Since $f \in \mathcal{R}$, $f \in \mathcal{R}'_{n_0}$. However, $f \in \mathcal{V}_{n_0}$ and thus $f \in \mathcal{R}_{n_0}$. Now, let f_n be sequence of C^2 -diffeomorphisms such that $f_n \rightarrow f$. Since $A_{n_0}(f)$ is horseshoe-like, the same holds for $A_{n_0}(f_n)$, for sufficiently large n . The reason is if $A_{n_0}(f_n)$ contains a local stable manifold $W_\delta^s(x_n, f_n)$ of some point $x_n \in A_{n_0}(f_n)$ then assuming $x_n \rightarrow x$, by the continuity of the stable manifolds, we have

$$W_\delta^s(x, f) = \lim_{n \rightarrow \infty} W_\delta^s(x_n, f_n) \subset \lim_{n \rightarrow \infty} A_{n_0}(f_n) = A_{n_0}(f),$$

which is a contradiction. By Corollary 3, $\text{Leb}(A_{n_0}(f_n)) = 0$ and hence,

$$\text{Leb}(A_{n_0}(f)) = \lim_{n \rightarrow \infty} \text{Leb}(A_{n_0}(f_n)) = 0.$$

□

3. HORSESHOE-LIKE PARTIALLY HYPERBOLIC ATTRACTOR OF POSITIVE VOLUME

The first example of a singular hyperbolic attractor was given by E. Lorenz in 70's during his study on the foundations of long range weather forecast. In fact, he gave an ODE equation that the numerical experiments showed the corresponding flow exhibits an attractor. Albeit the simplicity of the Lorenz equations (2 -degree polynomial), it was not a simple task to solve. There are two conceptual problems. The first one is the presence of a hyperbolic equilibrium accumulated by regular orbits which prevents the Lorenz attractor from being hyperbolic and the second is that the solutions slow down through the passage near the equilibrium, which means unbounded return times and thus unbounded integration errors. The impossibility of solving the equations leads Afraimovich-Bykov-Shil'nikov and Guckenheimer-Williams, independently (in the seventies), to propose a geometrical model. We illustrate in Figure 1 the schematic geometric model. Poincaré section Σ is showed by a square foliated by a contracting invariant foliation. The Poincaré section is partitioned by the two-dimensional stable manifold of the equilibrium into two rectangles whose images after coming back to the cross section are two triangles with curved sides as shown in the Figure 3. This model has a transitive attractor Λ with a dense set of periodic points.

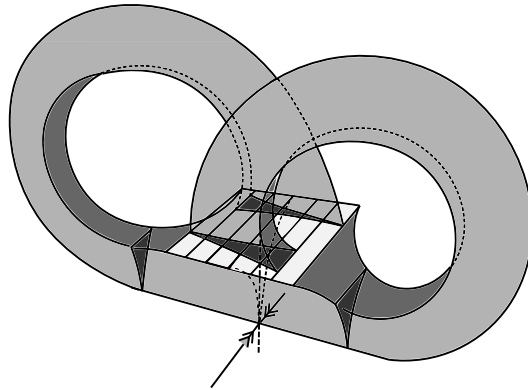


FIGURE 1. Geometric model

In this section an example of singular hyperbolic attractor with positive volume is given. The idea is to construct a fat horseshoe¹ H on the Poincaré section Σ of a geometric Lorenz attractor Λ such that $H \subset \Lambda \cap \Sigma$. Let us assume that we have constructed the fat horseshoe H . Then

$$H_\delta = \{X^t(x) : |t| \leq \delta, x \in H\},$$

is a subset of Λ and $\text{Leb}_3(H_\delta) \approx \delta \times \text{Leb}_2(H) > 0$. Therefore the geometric Lorenz attractor Λ has positive volume for its associated vector field and a priori for the time one map diffeomorphism. Note that the attractor Λ is horseshoe-like, i.e., it does not contain any local stable manifold, but it does contain an unstable manifold.

3.1. Fat triangulares on Σ . First, by a straightforward calculation, it is shown that fat triangulares on the cross section does not guarantee the positiveness of volume of the cross sectional attractor $\Lambda \cap \Sigma$ which consists of two Cantor cones.

Let $\Sigma = [-1, 1] \times [-1, 1]$ and $\Gamma = \{(x, y) \in \Sigma : x = 0\}$. For $k \geq 2$, define the function $F_k(x, y)$ on $\Sigma \setminus \Gamma$ as follows:

$$F_k(x, y) = \begin{cases} (2x^{1/2} - 1, 1/2(yx^{1/k} + 1)) & x > 0, \\ (-2|x|^{1/2} + 1, 1/2(y|x|^{1/k} - 1)) & x < 0. \end{cases}$$

We try to calculate the Lebesgue measure of the Cantor set $C(a)$ which is the intersection of the sectional attractor $\Lambda \cap \Sigma = \bigcap_{n \geq 0} F_k^n(\Sigma \setminus \Gamma)$ with segment $x = a$, see Figure 2. It can be seen

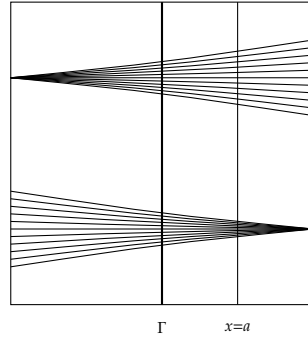


FIGURE 2. Sectional attractor (Cantor cones)

that

$$C(a) = \lim_{n \rightarrow +\infty} \bigcup_{x_0 \in f^{-n}(a)} F_k^n(x = x_0),$$

where

$$f(x) = \begin{cases} 2x^{1/2} - 1 & x > 0, \\ -2|x|^{1/2} + 1 & x < 0. \end{cases}$$

¹We call a horseshoe H fat if $\text{Leb}_2(H) > 0$.

Let $a_{0,1} = a$ and $b_0 = 1$. A simple calculation shows that

$$f^{-n}(a) = \left\{ \frac{a_{n,1}}{b_n}, \frac{a_{n,2}}{b_n}, \frac{a_{n,3}}{b_n}, \dots, \frac{a_{n,2^n}}{b_n} \right\},$$

where $a_{n,m} = (-1)^m(a_{n-1,\lceil m/2 \rceil} + (-1)^m b_{n-1})^2$ for $m = 1, 2, 3, \dots, 2^n$, and $b_n = 2^{2^{n+1}-2} = 4b_{n-1}^2$. Using the notation $A_{n,m} = F_k^n(x = \frac{a_{n,m}}{b_n})$, we have

$$\begin{aligned} \text{Leb}(C_n(a)) &= \sum_{m=1}^{2^n} \text{Leb}(A_{n,m}) = \sum_{\substack{1 \leq m \leq 2^n \\ m \text{ even}}} \text{Leb}(A_{n,m-1}) + \text{Leb}(A_{n,m}) \\ &= \sum_{\substack{1 \leq m \leq 2^n \\ m \text{ even}}} \frac{1}{2} \left(\left| \frac{a_{n,m-1}}{b_n} \right|^{1/k} + \left| \frac{a_{n,m}}{b_n} \right|^{1/k} \right) \text{Leb}(A_{n-1,m/2}) \\ &= \sum_{\substack{1 \leq m \leq 2^n \\ m \text{ even}}} \frac{1}{2} \left(\frac{(b_{n-1} - a_{n-1,m/2})^{2/k}}{4^{1/k} b_{n-1}^{2/k}} + \frac{(b_{n-1} + a_{n-1,m/2})^{2/k}}{4^{1/k} b_{n-1}^{2/k}} \right) \text{Leb}(A_{n-1,m/2}) \\ &= \sum_{m=1}^{2^{n-1}} \frac{1}{2^{1+2/k}} \left(\left(1 - \frac{a_{n-1,m}}{b_{n-1}}\right)^{2/k} + \left(1 + \frac{a_{n-1,m}}{b_{n-1}}\right)^{2/k} \right) \text{Leb}(A_{n-1,m}) \\ &\leq \sum_{m=1}^{2^{n-1}} \frac{1}{2^{2/k}} \text{Leb}(A_{n-1,m}) \\ &= \frac{1}{2^{2/k}} \text{Leb}(C_{n-1}) \end{aligned}$$

Therefore $\text{Leb}(C_n(a)) \leq \frac{1}{4^{n/k}} \text{Leb}(C_0(a)) = \frac{2}{4^{n/k}}$. Hence, $\text{Leb}(C(a)) = 0$. Then, Fubini theorem implies $\text{Leb}_2(\Lambda \cap \Sigma) = 0$.

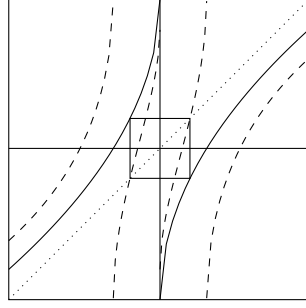
3.2. Modification of Lorenz map using Bowen's one. Now, we are trying to give an example of singular hyperbolic attractor with positive volume. We recall that $f : [-1, 1] \rightarrow [-1, 1]$ is a *Lorenz map* if it satisfies:

- $f(1) < 1$, $f(-1) > -1$ and $\lim_{x \rightarrow 0^\pm} f(x) = \mp 1$,
- $f'(x) > \alpha > 0$ for $x \neq 0$ and $\lim_{x \rightarrow 0^\pm} f'(x) = +\infty$.

Let f be an odd Lorenz map on $[-1, 1]$. Observe that the second iterate of Lorenz map, dashed graph in Figure 3, has two fixed points a and $-a$ and $f(a) = -a$. Choose $b \in (0, a)$ such that $f^2(b) = -a$. We assume $f(1) > -f(b)$. Now we follow the Bowen's construction of a fat horseshoe in [4]. For the sake of completeness, we provide the details here. Let $(\beta_n)_{n \geq 0}$ be a sequence of positive numbers with

$$\beta_0 = 2b, \quad \sum_{n=0}^{\infty} \beta_n < 2a \quad \text{and} \quad \beta_{n+1}/\beta_n \rightarrow 1.$$

Denote a word of length n with alphabet $\{0, 1\}$ by $w = w_1 w_2 \dots w_n$ where $w_i \in \{0, 1\}$ and $\ell(w) = n$. We allow the empty word $w = \emptyset$ of length 0 and $I_\emptyset = [-a, a]$. Let I_{w0} and I_{w1} be the

FIGURE 3. Lorenz map f and its second iterate

right and left intervals remaining after removing the interior of I_w^* from I_w where I_w^* is the closed interval of length $\beta_{\ell(w)}/2^{\ell(w)}$ and having the same center as I_w . Then

$$K = \bigcap_{n=0}^{\infty} \bigcup_{\ell(w)=n} I_w$$

is a Cantor set and $\text{Leb}(K) = 2a - \sum_{n=0}^{\infty} \beta_n > 0$. Now we replace the function f on $[f(b), -a] \cup [a, -f(b)]$ with new function and by abuse of notation, we continue to write f for the new function. Let $f : \bigcup_w f(I_{1w}^*) \rightarrow \bigcup_w I_w^*$ be a function such that

- $f|_{f(I_{1w}^*)}$ is a C^1 orientation preserving diffeomorphism and $f(f(I_{1w}^*)) = I_w^*$;
- $f'(f(x))f'(x) = 2$ for x an endpoint of I_{1w}^* ;
- $\sup_{x \in I_{1w}^*} |2 - f'(f(x))f'(x)| \rightarrow 0$ as $\ell(w) \rightarrow \infty$.

Next we can extend f continuously to $[f(b), -a]$ so that $f : [f(b), -a] \rightarrow [-a, a]$ is a C^1 diffeomorphism with $f'(f(x))f'(x) = 2$ for $x \in K \cap [b, a]$. The map f is defined on $[a, -f(b)]$ by $f(x) = -f(-x)$. Therefore, if we assume that $(f^2)'(a) = (f^2)'(b) = 2$, then f is a C^1 map such that $f^2 : [b, a] \rightarrow [-a, a]$ is a base map for a fat horseshoe in [4].

We can now define a Poincaré map $F(x, y) = (f(x), g(x, y))$ on the section $[-1, 1] \times [-1, 1]$. Notice that F is not defined on line $0 \times [-1, 1]$. The domain of F is partitioned like Figure 4, and g is a C^1 map such that

$$g(x, y) = \text{sgn}(x)f^{-1}(\text{sgn}(x)y)$$

for $(x, y) \in [-1, -b] \cup [b, 1] \times [f(b) - \epsilon, -f(b) + \epsilon]$, where $\epsilon = \frac{f(1)+f(b)}{2}$, $\text{sgn}(x) = |x|/x$ and f^{-1} is the inverse of right branch of f . In rest of the domain g is defined in a way that the image of these areas under F is shown in Figure 5. For $(x, y) \in [-a, -b] \cup [b, a] \times [-a, a]$, $F^2(x, y)$ is given by

$$F^2(x, y) = (f^2(x), -\text{sgn}(x)f^{-1}(-f^{-1}(\text{sgn}(x)y))).$$

Let A be the square $[-a, a] \times [-a, a]$, then

$$H = \bigcap_{k=-\infty}^{\infty} F^{2k}(A) = K \times K,$$

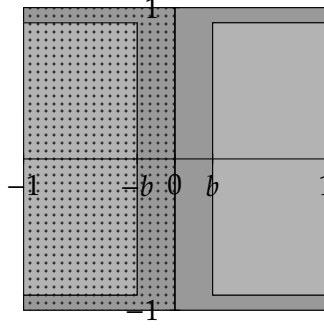


FIGURE 4. Domain partition of the Poincaré map

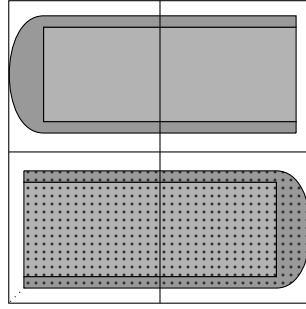


FIGURE 5. Image of the Poincaré map

and $\text{Leb}_2(H) = \text{Leb}(K)^2 > 0$. On the other hand,

$$H \subseteq \bigcap_{k=0}^{\infty} F^{2k}(A) \subseteq \bigcap_{k=0}^{\infty} F^k(\Sigma).$$

Last equality holds since $f^{2k}(\Sigma) \subseteq f^{2k-1}(\Sigma)$.

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DEPARTMENT OF MATHEMATICS, SHAHID BEHESHTI UNIVERSITY, TEHRAN 19839, IRAN

E-mail address: a.fakhari@sbu.ac.ir

INTERNATIONAL CENTER FOR THEORETICAL PHYSICS, STRADA COSTIERA 11, 34100 TRIESTE, ITALY

E-mail address: msoufin@gmail.com